

The quaternionic commutator bracket and its implications

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Abstract

A quaternionic commutator bracket for position and momentum shows that the quaternionic wave function, *viz.* $\tilde{\psi} = (\frac{i}{c}\psi_0, \vec{\psi})$, represents a state of a particle with orbital angular momentum, $L = 3\hbar$, resulting from the internal structure of the particle. This angular momentum can be attributed to spin of the particle. The vector $\vec{\psi}$, points along the direction of \vec{L} . When a charged particle is placed in an electromagnetic fields the interaction energy reveals that the magnetic moments interact with the electric and magnetic fields giving rise to terms similar to Aharonov-Bohm and Aharonov-Casher effects.

1 Introduction

In quantum mechanics particles are described by relativistic or non-relativistic wave equations. Each equation associates a spin state of the particle to its wave equation. For instance, Schrodinger equation applies to the spin-0 particles in the non-relativistic domain, while the appropriate relativistic equation for spin-0 particles is the Klein-Gordon equation. The relativistic spin-1/2 particles are governed by the Dirac equation. Recall that this equation reduces, in the non-relativistic limit, to the Schrodinger - Pauli equation [2, 1]. Particles with spin -1 are described by Maxwell equations. Charged particles interact with the electromagnetic field. The effect of the interaction of these particles is obtained via the *minimal coupling* ansatz, where the momentum of the free particle (p_μ) is replaced by $p_\mu + qA_\mu$, where A_μ is the photon field. With this prescription, Dirac equation as well as the Schrodinger -

Pauli equation predict the existence of spin for the electron through the interaction of its magnetic moment with magnetic field present. Thus, the spin angular momentum is deemed to be a quantum phenomenon having no classical analogue. Hence, the spin is an intrinsic property of a quantum particle. Therefore, the spin is not a result of the rotation of a point quantum particle (like the electron). It could be associated with the space in which the particle is defined. In our new formulation the particle is defined by the scalar and vector wavefunctions. The spin angular momentum can be defined in terms of these vector and scalar wavefunctions.

In a recent paper, we have unified the above three quantum wave equations in a single equation. We call this equation a unified quantum wave equation [3]. In the present work, we would like to investigate the nature of spin relying on our unified quantum wave equation, and using the quaternionic commutator bracket between position and momentum, *viz.*, $[x_i, p_j] = i\hbar \delta_{ij}$ [2]. This bracket is a fundamental cornerstone in formulating quantum mechanics.

Such a generalization led to interesting physical results pertaining to the nature of the wave equation describing quantum particles. We have found that an intrinsic angular momentum, related to its internal nature a quantum particle, is associated with the quaternionic wave function $\tilde{\psi}$ that we recently found. It is generally understood that spin is not due to the point-particle rotation. This is true if we treat a particle as a point particle. But, we have recently shown that the wavy nature of a quantum particle due to its wavepacket nature undergoes an internal rotation [3]. A wavepacket consists of two waves moving in opposite directions with speed of light. If a quantum particle is thought of a wavepacket or an extended object, then an internal rotation is plausible.

Our current investigations revealed that the scalar (ψ_0) and vector ($\vec{\psi}$) wavefunctions describe a particle with internal orbital angular momentum, $L = \hbar$; and that the vector $\vec{\psi}$ is directed along the direction of \vec{L} . This internal angular momentum can be attributed to some kind of spin. These may define the helicity states of the quantum particle. Such states can be compared with spinor representation of Dirac equation. When a charged particle is placed in an electromagnetic field, the magnetic moments interact with both electric and magnetic fields giving rise to Aharonov-Bohm and Aharonov-Casher effects.

2 The fundamental commutator bracket

The position and momentum commutation relation in quaternionic now reads

$$[\tilde{X}, \tilde{P}] \tilde{\psi} = i\hbar \tilde{\psi}, \quad (1)$$

where

$$\tilde{X} = (ict, \vec{r}), \quad \tilde{P} = \left(i\frac{E}{c}, \vec{p}\right), \quad \tilde{\psi} = \left(\frac{i}{c}\psi_0, \vec{\psi}\right). \quad (2)$$

The multiplication of two quaternions, $\tilde{A} = (a_0, \vec{a}), \tilde{B} = (b_0, \vec{b})$ is given by

$$\tilde{A}\tilde{B} = (a_0b_0 - \vec{a} \cdot \vec{b}, a_0\vec{b} + \vec{a}b_0 + \vec{a} \times \vec{b}). \quad (3)$$

Using eq.(3) and the fact that in quantum mechanics, $\vec{p} = -i\hbar\vec{\nabla}$ and $E = i\hbar\frac{\partial}{\partial t}$, and eq.(1) to get

$$\vec{L} \cdot \vec{\psi} = \frac{3\hbar}{c} \psi_0, \quad (4)$$

$$\vec{L} \psi_0 = 3\hbar c \vec{\psi}, \quad (5)$$

and

$$\vec{L} \times \vec{\psi} = 0. \quad (6)$$

Equation (5) states \vec{L} transform the scalar wavefunction ψ_0 into the vector wavefunction $\vec{\psi}$. Notice however that in quantum mechanics, $\vec{L} \times \vec{L} = i\hbar\vec{L}$. But according to eq.(5) and (6), $\vec{L} \times \vec{L} = 0$. Thus, the physical meaning of ψ has now become clear. We have recently developed the quaternionic quantum mechanics but the physical meaning of the quaternion wavefunction remained unsorted [6, 3, 7].

Now take the dot product of eq.(5) with \vec{L} and use eq.(4) to get

$$L^2 \psi_0 = 9\hbar^2 \psi_0. \quad (7)$$

Similarly, multiplying eq.(4) by \vec{L} (from right) and using eq.(5) and, the vector identity $\vec{A} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{A} \cdot \vec{B}) - A^2 \vec{B}$, we obtain

$$L^2 \vec{\psi} = 9\hbar^2 \vec{\psi}. \quad (8)$$

Hence, the quaternion components, ψ_0 and $\vec{\psi}$ represent a state of a particle with a total orbital angular momentum of

$$L = 3\hbar. \quad (9)$$

This is a quite interesting result. It seems that this angular momentum arises from an internal degree of freedom. It may result from a rotation of some internal structure of the particle. In Dirac's theory the spin of the electron doesn't emerge from the equation to be 1/2, but is deduced to be 1/2 from the way the electron interacts with the photon field.

We have shown recently that the unified quantum wave equations are [7]

$$\vec{\nabla} \cdot \vec{\psi} - \frac{1}{c^2} \frac{\partial \psi_0}{\partial t} - \frac{m_0}{\hbar} \psi_0 = 0, \quad (10)$$

$$\vec{\nabla} \psi_0 - \frac{\partial \vec{\psi}}{\partial t} - \frac{m_0 c^2}{\hbar} \vec{\psi} = 0, \quad (11)$$

and

$$\vec{\nabla} \times \vec{\psi} = 0. \quad (12)$$

In the ordinary quantum mechanics, a particle is described by a scalar or spinor, however, a particle is now described by a scalar and a vector. In all a particle is described by a four component function. In electromagnetism, the electromagnetic fields \vec{E} and \vec{B} are vector field. But at the fundamental level these two field are represented by a scalar field φ and a vector field \vec{A} , respectively. In this manner, at the fundamental level, a particle should be described by some similar fields, which are here ψ_0 and $\vec{\psi}$. This makes the analogy between the field and particle representations symmetric. The electromagnetic wave is transverse, i.e., $\vec{E} \perp \vec{B} \perp \vec{k}$, while a particle wave is longitudinal. In our formulation, this feature is very clear. Equation (6) states that the spin direction is along the direction of the field $\vec{\psi}$. As for photons, which are described by their polarization a quantum particle should have some similar ansatz, where the spin is directly associated with the particle wavefunction. Thus, the wavefunction incorporates the spin states. In Dirac theory the spin is deduced from the interaction of the electron with the photon magnetic field. In Schrodinger-Pauli theory the spin of the electron is also deduced from their equation in the way the electron spin is coupled to the photon magnetic field. In ordinary quantum mechanics, the spin of the particle does not emerge from its wavefunction, but eqs.(7) and (8) show that it does in our present formulation.

And if $\psi_0 = -\vec{v} \cdot \vec{\psi}$, i.e., when $\vec{\psi}$ is projected along the direction of motion, then [8]

$$i \hbar \frac{d\psi_0}{dt} = m_0 c^2 \psi_0, \quad i \hbar \frac{d\vec{\psi}}{dt} = m_0 c^2 \vec{\psi}. \quad (13)$$

Now differentiate eq.(5) with respect to time and use eq.(13) to obtain

$$\frac{d\vec{L}}{dt} = 0. \quad (14)$$

this implies that the orbital angular momentum is a constant of motion. It is thus a conserved quantity. Moreover, if we take the dot product of eq.(11) with \vec{L} and use eqs.(4), (5) and (10) then

$$\frac{\partial \vec{L}}{\partial t} \cdot \vec{\psi} = 0. \quad (15)$$

and if we use the fact that

$$\frac{d\vec{L}}{dt} = \frac{\partial \vec{L}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{L}, \quad (16)$$

then eq.(15) implies that

$$\frac{d\vec{L}}{dt} \cdot \vec{\psi} = 0. \quad (17)$$

Thus, either \vec{L} is conserved or the external torque, $\vec{\tau} = \frac{d\vec{L}}{dt}$, is perpendicular to $\vec{\psi}$. Thus, the internal rotation of the particle is due to the self-interaction (internal constituents) of the particle.

Now apply the condition, $\psi_0 = -\vec{v} \cdot \vec{\psi}$, in eqs.(4), (5) and (6), and using eq.(9), yield

$$\hat{L} \cdot \vec{v} = -c, \quad \hat{L} = \frac{\vec{L}}{L}. \quad (18)$$

This shows that \vec{L} is along the opposite direction of motion and that the particle moves with speed of light.

Now let us choose

$$\vec{\psi} = \frac{1}{c} \vec{\sigma} \psi_0, \quad (19)$$

where σ are the Pauli matrices. In this case, $\vec{\psi}$ would represent a spin vector wave. Apply eq.(18) in eqs.(5) and (6) to obtain

$$\vec{L} = 3\hbar \vec{\sigma}. \quad (20)$$

If substitute eq.(19) in the condition $\psi_0 = -\vec{v} \cdot \vec{\psi}$, we will obtain

$$\vec{\sigma} \cdot \frac{\vec{v}}{c} = -1. \quad (21)$$

This shows that the spin component is antiparallel to the direction of motion. This case agrees with that in eq.(18). Hence, the particle is left-handed! It seems that there is some internal degree of freedom (angular momentum) associated with quaternionic particles. Or alternatively, that the quaternionic space has some twisting properties.

3 Interaction energy

Let us now consider the interaction energy (U) of the magnetic dipoles in the presence of electromagnetic fields. This energy is due to spin (\vec{S}) and orbital angular momentum (\vec{L}) and the corresponding moments associated with them, $\vec{\mu}_s$ and $\vec{\mu}_\ell$. The quaternionic form of the total angular momentum is defined as

$$\tilde{J} = (0, \vec{L} - i\vec{S}). \quad (22)$$

The corresponding interaction energy can be written as [9]

$$\tilde{U} = (iU, \vec{\tau}) = -\tilde{\mu}\tilde{F}, \quad (23)$$

where

$$\tilde{\mu} = (0, \vec{\mu}_\ell - i\vec{\mu}_s), \quad \tilde{F} = (0, \frac{\vec{E}}{c} + i\vec{B}). \quad (24)$$

When the dipole magnetic moment is placed in an external magnetic field, it experiences a torque ($\vec{\tau}$). The torque tends to align the dipole with the field. But when the interaction energy is constant, the dipole moment precesses with the magnetic field. Substituting eq.(24) in eq.(23) and equating the real and imaginary parts of the resulting equations yield

$$U = -\vec{\mu}_\ell \cdot \frac{\vec{E}}{c} - \vec{\mu}_s \cdot \vec{B}, \quad \vec{\mu}_\ell \cdot \vec{B} = \vec{\mu}_s \cdot \frac{\vec{E}}{c}, \quad \vec{\tau} = \vec{\mu}_s \times \vec{B} + \vec{\mu}_\ell \times \frac{\vec{E}}{c}, \quad \vec{\mu}_\ell \times \vec{B} = \vec{\mu}_s \times \frac{\vec{E}}{c}. \quad (25)$$

Equation (25) encompasses all possible terms that may arise due to the presence of $\vec{\mu}_s$ and $\vec{\mu}_\ell$. The interaction of a magnetic moment with electric field has not been considered widely by physicists [10, 11], and hence this term is deemed to vanish. This term violates parity and time reversal invariance but respects rotational invariance [10]. Owing to duality between electric and magnetic fields, we trust such terms should be present. It is shown by Aharonov-Casher that a phase shift occurs for a neutral particle with a nonzero magnetic dipole moment moving in an electric field [11]. This is given by

$$\Delta\varphi_{sE} = \oint (\vec{\mu}_s \times \vec{E}) \cdot d\vec{r}. \quad (26)$$

Using eq.(25) this is transformed into

$$\Delta\varphi_{\ell B} = \oint (c\vec{\mu}_\ell \times \vec{B}) \cdot d\vec{r}. \quad (27)$$

This states that the effect of a spin magnetic moment in an electric field is equivalent to an orbital magnetic moment in a magnetic field. Aharonov-Bohm also showed that in the absence of electric field in the region, the phase

shift of the particle wavefunction is given by

$$\Delta\varphi = -qV\Delta t/\hbar, \quad (28)$$

where V is the electric potential and Δt is the time spent in the electric potential [12]. In magnetic Aharonov the phase shift is given by

$$\Delta\varphi = q\phi_B/\hbar, \quad (29)$$

where ϕ_B is the magnetic flux enclosed by the solenoid. The magnetic flux can be written as

$$\phi_B = \int \vec{B} \cdot d\vec{A}, \quad (30)$$

but if

$$\frac{d\vec{A}}{dt} = \frac{\vec{L}}{2m} = \text{const.}, \quad \vec{A} = \frac{\vec{L}}{2m} \Delta t, \quad (31)$$

reflecting conservation of the orbital angular momentum in the closed loop, then applying eqs.(30) and (31) in eq.(29) yields

$$\Delta\varphi = \frac{qt}{2\hbar m} \vec{L} \cdot \vec{B} = \frac{1}{\hbar} \vec{\mu}_\ell \cdot \vec{B} \Delta t. \quad (32)$$

Comparing eqs.(28) and (32) reveals that

$$\frac{d\varphi}{dt} = \frac{qV}{\hbar}. \quad (33)$$

Hence, in the absence of electric field, the charge behaves as a magnetic dipole interacting with a magnetic field. We thus may attribute the effect above to the spin and orbital angular momentum that the charge carries. It is of importance to remark that the rate of change phase in the Josephson junction, that is mediated by Cooper pairs, amounts to double the value in eq.(33) [13].

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